Math 250
Fall 2013
Midterm 1 - Form A
10/07/13

Name (Print): ____________________

This exam contains 5 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	25	
4	10	
5	15	
6	15	
7	15	
Total:	100	

1. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 6 & 8 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 & 5 \\ 6 & 2 & 9 \end{bmatrix}$$

Compute the following. If it's not possible write impossible.

(a) A^T + B

Ans:

$$\begin{bmatrix} 5 & 4 & 11 \\ 8 & 7 & 17 \end{bmatrix}$$

(b) $B^T + A^T$

Ans: Not possible

(c) AB

Ans:

$$\begin{bmatrix} 16 & 4 & 23 \\ 46 & 10 & 65 \\ 72 & 16 & 102 \end{bmatrix}$$

(d) $B^T A^T$

Answer: (Note that $B^T A^T = (AB)^T$)

$$\begin{bmatrix} 16 & 46 & 72 \\ 4 & 10 & 16 \\ 23 & 65 & 102 \end{bmatrix}$$

(e) B^TA

Ans: Not possible

2. (10 points) Determine if the given set is linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Ans: The REF is

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

and the RREF is the identity, so the set is linearly independent.

3. (a) (15 points) Find the Reduced Row Echelon Form (RREF) of

$$A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & -4 & 1 \\ -1 & 1 & 1 & -2 \\ 2 & -1 & 3 & 1 \end{bmatrix}$$

Ans:

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (5 points) Write the vector form of the general solution to $Ax = \mathbf{0}$, where A is the matrix in part a.

$$\begin{bmatrix} -4x_3 \\ -5x_3 \\ x_3 \\ 0 \end{bmatrix}$$

(c) (5 points) Are the columns of A linearly dependent? Yes, as can be seen from the RREF.

4. (10 points) A pharmaceutical is producing three types of supplements that provides various portion of vitamins B_1, B_6, B_{12}.

	Supplement A	Supplement B	Supplement C
B1	1 mg	1 mg	2 mg
B6	1 mg	2 mg	2 mg
B12	2 mg	3 mg	4 mg

A patient needs to consume exactly 6 mg of B_1, 8 mg of B_6 and 14 mg of B_{12} in a meal. What is the maximum amount of Supplement C can he take to provide exactly the required amounts of each of the three vitamins?

Ans: The augmented matrix to look at is

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 1 & 2 & 2 & 8 \\ 2 & 3 & 4 & 14 \end{bmatrix}$$

and its REF is

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore $B = 2$. That leads to $A + 2C = 4$. So the maximum amount C can be is 2 (when $A = 0$.)

5. (15 points) Determine, if possible, a value of r for which the given set is **linearly dependent**.

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ r \end{bmatrix}, \right\}$$

Ans: The REF of the matrix

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 3 & r \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 3 & r \\ 0 & 1 & \frac{2r-1}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Note that all columns of the REF has a pivot. So the set is linearly independent. In other words, there is no value of r that can make the set linearly independent.

Alternative answer:

1. It is clear that the first two vectors in the set are not parallel.

2. It is also clear that the third vector cannot be written as a linear combination of the first two because the 2nd entry of the first two vectors are 0 while the 2nd entry of the last vector is 1.

Therefore the set is linearly independent. You must mention **both** of these items to get this answer correct.

6. In the following, determine if the matrix A is invertible or not. If A is not invertible, explain why. If A is invertible, write down the inverse of A.

   (a) (5 points)

$$\begin{bmatrix} -1 & 5 \\ 3 & -1 \end{bmatrix}$$

   Yes. Its inverse is:

$$\begin{bmatrix} \frac{1}{14} & \frac{5}{14} \\ \frac{3}{14} & \frac{1}{14} \end{bmatrix}$$

   (b) (5 points)

$$\begin{bmatrix} 1 & 5 & 9 \\ 3 & 15 & 7 \\ 4 & 20 & 8 \end{bmatrix}$$

   Not invertile, since the first and second columns are linearly dependent.

   (c) (5 points)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

   Yes. It is an elementary matrix representing the row operation of swapping the 2nd and 3rd rows. It is its own inverse.

7. Let

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}; B^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; C^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Compute the following

(a) (5 points) $(AB^T)^{-1}$

Ans:

$(AB^T)^{-1} = (B^T)^{-1}A^{-1} = (B^{-1})^T A^{-1}$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(b) (5 points) $(B^T A^T)^{-1}$

Ans: $(B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1} = (A^{-1})^T(B^{-1})^T$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(c) (5 points) $(AC^T B)^{-1}$

Ans: $(AC^T B)^{-1} = B^{-1}(C^T)^{-1}A^{-1} = B^{-1}(C^{-1})^T A^{-1}$

$$\begin{aligned} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ = & \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\ = & \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 3 \\ 2 & 0 & 4 \end{bmatrix} \end{aligned}$$